Worcester County Mathematics League

Varsity Meet 3
January 25, 2012

Coaches' Copy
Rounds, Answers and Solutions

Worcester County Mathematics League

Varsity Meet 3 – January 25, 2012

Round 1: Similarity and Pythagorean Theorem

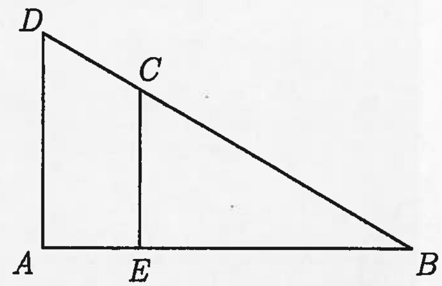
1

All answers must be in simplest exact form in the answer section

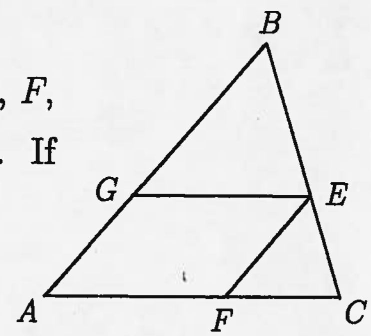
NO CALCULATOR ALLOWED

Note that the diagrams are not drawn to scale

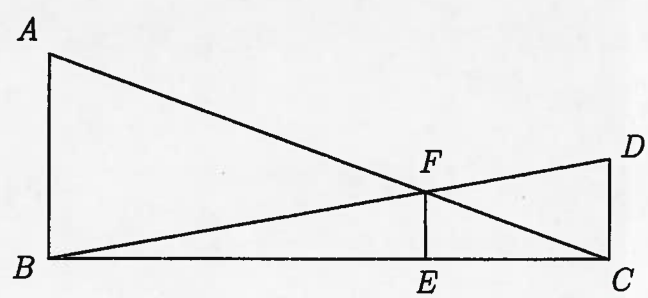
1. The diagram shows right triangles ABD and EBC with right angles $\angle A$ and $\angle CEB$, respectively. If $BE = 4$, $BC = 5$ and $AE = \frac{1}{3}BE$, find the exact length of $\overline{BD}$.



2. The diagram to the right shows triangle ABC and points E, F, and G such that $EC : BC = 1 : 3$, $\overline{EF} \parallel \overline{AB}$ and $\overline{GE} \parallel \overline{AC}$. If the area of triangle ABC is 144, compute the area of quadrilateral $AGEF$.



3. In the diagram below $\overline{AB} \parallel \overline{FE} \parallel \overline{DC}$ and $\overline{AB} \perp \overline{BC}$. If $AB = 24$, $DC = 8$ and $BE = 48$, compute the sum $FE + EC$.



ANSWERS

(1 pt.) 1.____________________

(2 pts.) 2.____________________

(3 pts.) 3.____________________

Shrewsbury, AMSA Charter, Worc. Academy

Worcester County Mathematics League

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Round 2: Algebra 1 - Open

2

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

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1. This coming Halloween Tom plans to scare twice as many people as Sam, and Sam plans to scare three times as many people as Roz. All together they plan to scare at most 2012 people. If no one is scared more than once, at most how many people does Sam plan to scare?

2. Two cars race on a 4-mile oval track. The sum of the speeds at which the cars travel is 200 miles per hour. If the faster car gains 1 lap after 40 minutes of racing, compute the speed of the slower car (in miles per hour).

3. Two jars contain an equal number of marbles. All of the marbles are either blue or green. In the first jar, the ratio of blue to green marbles is 7:1. In the second jar, the ratio of blue to green marbles is 9:1. If there are 90 green marbles altogether, how many blue marbles are in the second jar?

ANSWERS

(1 pt.) 1.________________________ people

(2 pts.) 2. ________________________ miles per hour

(3 pts.) 3.________________________blue marbles

Shepherd Hill, Bancroft, Algonquin

Worcester County Mathematics League

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Round 3: Functions

3

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. Let f and g be invertible functions such that $f(1)=3$, $f(2)=7$, $f(3)=8$, $g(2)=-1$, $g(6)=2$, and $g(7)=3$. Find the value of $f^{-1}(g^{-1}(3))$.

2. For any real number x the function $h(x)$ satisfies the equation $2h(x)+h(1-x)=x^2$. Compute the value of $h(5)$.

3. Suppose $f(0)=1$ and $f(1)=0$. For any integer $n \geq 1$, $f(2n)=f(n)+1$ and $f(2n+1)=f(n)-1$. Compute the value of $f(2012)$.

ANSWERS

(1 pt.) 1. ______________________

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________

Algonquin, Westborough, Hudson

Worcester County Mathematics League

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Round 4: Combinations and Permutations

4

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

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1. In how many different ways can two white chips and seven red chips be arranged around a circle? Assume that other than their color the chips are identical.

2. Team #1 from Springfield's math team consists of 3 girls and 2 boys. If three members of the team were chosen to compete in Round #1 and at least 2 of them were girls, in how many different ways could the members of the team have been chosen to compete in Round #1? Assume that the order they are chosen to compete does not matter.

3. After today's math meet several mathletes are going to make 3-scoop ice cream sundaes. In how many ways can a three-scoop sundae be made if there are 20 flavors of ice cream available and each flavor can be used more than once? Assume that the order that the scoops are put into the sundae does not matter.

<u>ANSWERS</u>

(1 pt.) 1.____________________

(2 pts.) 2.____________________

(3 pts.) 3.____________________

St. John's, Tahanto, Shrewsbury

Worcester County Mathematics League

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Round 5: Analytic Geometry

5

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

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1. Find the area of the triangle whose vertices are the center and y-intercepts of the circle $(x-3)^2+(y+6)^2=25$.

2. Line l_1 passes through the midpoint of the line segment whose endpoints are (1, 6) and (5, -2). Line l_2 passes through the points (-9, 1) and (6, -4). Also, line l_1 is perpendicular to line l_2. If line l_1 contains the point (6, b), compute the value of b.

3. Find all of the points of intersection of the circle whose equation is $x^2-12x+y^2-20y+120=0$ with the asymptotes of the hyperbola whose equation is $x^2-y^2=9$.

ANSWERS

(1 pt.) 1.________________________

(2 pts.) 2. ________________________

(3 pts.) 3. ________________________________

South, AMSA Charter, St. Peter-Marian

Worcester County Mathematics League

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TEAM ROUND

All answers must *either* be in simplest exact form *or* as decimals rounded correctly to at least three decimal places, unless stated otherwise (2 pts. each)

APPROVED CALCULATORS ALLOWED

1. The parabola $y = ax^2 + bx + c$ passes through the x-intercepts of the parabola $y = x^2 - 4x - 12$ and has its vertex at the center of the conic $x^2 - 4x + y^2 - 8y + 10 = 0$. Find the sum $a + b + c$.

2. X, Y and Z are three different digits from 1 to 9 (inclusive) forming the number XYZ. For example, the digits 2, 6, and 4 would form the number 264. Find the smallest possible value of $\dfrac{XYZ}{X+Y+Z}$.

3. For how many of the following numbers is 144 a factor? Notice that the list of numbers having the established pattern is typed on two lines.
   1!, 2! + 3!, 3! + 4! + 5!, 4! + 5! + 6! + 7!, 5! + 6! + 7! +8! +9!,...
   ..., 100! + 101! + 102! +...+199!

4. There are N dimes in a jar. Consider the following 5 statements about how many dimes there are in the jar:
   - i. The number of dimes is a prime number
   - ii. There are at least 7 dimes
   - iii. There are less than 17 dimes
   - iv. The number of dimes is a multiple of 3
   - v. The number of dimes is odd

   What is the sum of all of the possible values of N such that exactly 4 of the above statement are true?

5. Let p, q, r, s, and t be whole numbers representing the degree measures of the five angles of a convex pentagon. If $p < q < r < s < t$ and p, q, r, s, t form an arithmetic sequence, how many possible values are there for p?

6. Determine the total number of right triangles in which the lengths of all of the sides are integers and one side has length 20.

7. Let x and y be real numbers. Determine the smallest possible value of the expression $|2-x| + |x-y| + |y-2012|$.

8. Let $f(x) = x + \lfloor x \rfloor$, where $\lfloor x \rfloor$ represents the greatest integer less than or equal to x. Compute the value of $f^{-1}(2.5)$.

9. Of all of the integers from 1 to 1000 (inclusive), how many contain at least one odd digit?

St. John's, QSC, Marlboro, Worc. Academy (4 & 6), Hudson, Shepherd Hill, Quaboag (8 & 9)

Worcester County Mathematics League

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ANSWER SHEET – TEAM ROUND

All answers must *either* be in simplest exact form *or* as decimals rounded correctly to at least three decimal places unless stated otherwise

(2 pts. each)

1.______________________________

2.______________________________

3.______________________________

4. ______________________________

5. ______________________________

6.______________________________

7.______________________________

8.______________________________

9.______________________________

Worcester County Mathematics League

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ANSWERS

Round 1

1. $\frac{20}{3} = 6\frac{2}{3} = 6.\overline{6}$
2. 64
3. 22

Round 2

1. 603
2. 97
3. 360

Round 3

1. 2
2. $\frac{34}{3} = 11\frac{1}{3} = 11.\overline{3}$
3. –4

Round 4

1. 4
2. 7
3. 1540

Round 5

1. 12
2. 11
3. (6, 6) and (10, 10)
   Need both ordered pairs in either order

Team Round

1. $\frac{15}{4} = 3\frac{3}{4} = 3.75$
2. $\frac{21}{2} = 10\frac{1}{2} = 10.5$
3. 96
4. 58
5. 35
6. 4
7. 2010
8. $\frac{3}{2} = 1\frac{1}{2} = 1.5$
9. 876

Worcester County Mathematics League

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BRIEF SOLUTIONS

Round 1

1. $AE = \frac{1}{3}BE \Rightarrow AB = \frac{1}{3}BE + BE \Rightarrow AB = \frac{4}{3}BE \Rightarrow \frac{AB}{BE} = \frac{4}{3}$. By similar triangles $\frac{BD}{BC} = \frac{4}{3} = \frac{x}{5} \Rightarrow x = \frac{20}{3}$.

2. First note that $\Delta GBE \sim \Delta ABC \sim \Delta FEC$ by AA Similarity. So, the ratio of the area of ΔFEC to the area of ΔABC is $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$ because $\frac{EC}{BC} = \frac{1}{3}$. Likewise the ratio of the area of ΔGBE to the area of ΔABC is $\left(\frac{2}{3}\right)^2 = \frac{4}{9}$ because $\frac{BE}{BC} = \frac{BC - EC}{BC} = 1 - \frac{1}{3} = \frac{2}{3}$. Finally, the area of AGEF is the equal to the quantity

(area of ΔABC) – (area of ΔGBC) – (area of ΔFEC) = area of $\Delta ABC \cdot \left(1 - \frac{4}{9} - \frac{1}{9}\right) = \frac{4}{9}$(area of ΔABC)

$= \frac{4}{9} \cdot 144 = 64$.

3. By similar triangles $\frac{8}{EC+48} = \frac{FE}{48}$ and $\frac{24}{EC+48} = \frac{FE}{EC}$. Dividing the second equation by the first yields

$3 = \frac{\frac{FE}{EC}}{\frac{FE}{48}} \Rightarrow 3 = \frac{48}{EC} \Rightarrow EC = 16$. Now, using substitution with the first equation we have $\frac{8}{16+48} = \frac{FE}{48}$

$\Rightarrow FE \cdot 64 = 8 \cdot 48 \Rightarrow FE = 6$. Therefore, $FE + EC = 6 + 16 = 22$.

Round 2

1. Let $n =$ the number of people Roz plans on scaring, so that $3n$ and $6n =$ the number of people that Sam and Tom plan on scaring, respectively. We have $n + 3n + 6n = 10n \le 2012 \Rightarrow n \le 201.2 \Rightarrow n$ is at most 201. Therefore, Sam plans on scaring at most $3(201) = 603$ people.

2. Let $R =$ the speed of the slower car, so that $200 - R =$ the speed of the faster car. In 40 minutes $= \frac{2}{3}$ of an hour the faster car will have traveled exactly 4 miles more than the slower car. We have:

$\frac{2}{3}R + 4 = \frac{2}{3}(200 - R) \Rightarrow 2R + 12 = 400 - 2R \Rightarrow 4R = 388 \Rightarrow R = 97$.

3. Suppose that there are n marbles in each jar. Then, because the ratio of blue to green marbles in the first jar is 7:1, there are $\frac{1}{8}n$ green marbles in the first jar. Likewise, because the ratio of blue to green marbles in the second jar is 9:1, there are $\frac{1}{10}n$ green marbles in the second jar. Thus, the total number of green marbles is

$\frac{1}{8}n + \frac{1}{10}n = \frac{18}{80}n = \frac{9}{40}n = 90 \Rightarrow n = 400$. Therefore, the number of blue marbles in the second jar is

$\frac{9}{10}n = \frac{9}{10} \cdot 400 = 360$.

Round 3

1. First, $g(7)=3 \Rightarrow g^{-1}(3)=7$. Next, $f(2)=7 \Rightarrow f^{-1}(7)=2$ so that $f^{-1}(g^{-1}(3))=f^{-1}(7)=2$.

2. If $x=5$ we have $2f(5)+f(-4)=25$. If $x=-4$ we have $2f(-4)+f(5)=16$. Solving these equations simultaneously yields $f(5)=\frac{34}{3}$ (and $f(-4)=\frac{7}{3}$).

3. Here's the long way: Using the first definition when the argument of the function is even and using the second definition when the function is odd gives the following chain of equivalencies:
$f(2012)=f(2\cdot 1006)=f(1006)+1=f(2\cdot 503)+1=f(503)+1+1$
$=f(2\cdot 251+1)+2=f(251)-1+2=f(2\cdot 125+1)+1=f(125)-1+1=f(2\cdot 62+1)=f(62)-1=f(2\cdot 31)-1$
$=f(31)+1-1=f(2\cdot 15+1)=f(15)-1=f(2\cdot 7+1)-1=f(7)-1-1=f(2\cdot 3+1)-2=f(3)-1-2$
$=f(2\cdot 1+1)-3=f(1)-1-3=-4$. Of course it is easier to realize that upon successively dividing the argument of the function by 2 and truncating at the decimal point, the even arguments provide +1 for the function and odd arguments provide -1 for the function. For $f(2012)$ we have the sequence 2012, 1006, 503, 251, 125, 62, 31, 15, 7, 3, 1. Therefore, we have $f(2012)=1+1-1-1-1+1-1-1-1-1+0=-4$.

Round 4

1. The two white chips can be next to each other, have one red chip between them, two reds chips between them, or three red chips between them. So there are 4 ways.

2. Call the members G_1, G_2, G_3, B_1, B_2 (representing each of the three girls and the two boys). If B_1 is selected to compete, then there are 3 ways the girls can be chosen: G_1, G_2, B_1, G_1, G_3, B_1, and G_2, G_3, B_1. Likewise, if B_2 is selected to compete, then there are 3 ways the girls can be chosen: G_1, G_2, B_2, G_1, G_3, B_2, and G_2, G_3, B_2. Finally, if neither boy is chosen, there is one way to select the girls: G_1, G_2, G_3. Therefore, there are a total of 7 ways to select the 3 members to compete in the round.

3. Here's one way using 3 cases. First, there are 20 ways for all three scoops to be the same. Next, there are $20\cdot 19\cdot 1=380$ ways for exactly two scoops to be the same (and the other be different). Finally, there are ${}_{20}C_3=\frac{20!}{3!\,17!}=\frac{20\cdot 19\cdot 18}{6}=20\cdot 19\cdot 3=1140$ ways for all three scoops to be different. Therefore there are $20+380+1140=1540$ different ways that the sundaes can be made.

Round 5

1. For the circle $(x-3)^2+(y+6)^2=25$, the center is (3, -6). The y-intercepts are the solutions to the equation $9+(y+6)^2=25 \Rightarrow (y+6)^2=16 \Rightarrow y+6=\pm 4 \Rightarrow y=-2,-10$. Therefore, the base of the triangle is 8 and the corresponding altitude has length 3. The area of the triangle is $\frac{1}{2}\cdot 8\cdot 3=12$.

2. The midpoint of (1, 6) and (5, -2) is (3, 2). The line passing through (-9, 1) and (6, -4) has slope $\frac{1-(-4)}{-9-6}=\frac{5}{-15}=-\frac{1}{3} \Rightarrow$ the slope of line l_1 is 3. So, the point-slope equation for line l_1 is $y-2=3(x-3)$. This line passes through the point (6, b) so that $b-2=3(6-3) \Rightarrow b-2=9 \Rightarrow b=11$.

3. The asymptotes of the hyperbola $x^2 - y^2 = 9 \Rightarrow \frac{x^2}{9} - \frac{y^2}{9} = 1$ are $y = \pm\frac{9}{9}x = \pm x$. Solving $y = \pm x$ simultaneously with the equation for the other conic will give us their points of intersection. First, if $y = x$ and using substitution we have $x^2 - 12x + x^2 - 20x + 120 = 0 \Rightarrow x^2 - 16x + 60 = 0 \Rightarrow (x-10)(x-6) = 0$ $\Rightarrow x = 10$ or $x = 6$. These x-values correspond to the points (10, 10) and (6, 6). Next, if $y = -x$, by substitution we have $x^2 - 12x + x^2 + 20y + 120 = 0 \Rightarrow x^2 + 4x + 60 = 0$. The roots of this equation are complex because the equation's discriminant is < 0 ($4^2 - 4 \cdot 1 \cdot 60 = -244$). Therefore there are only two points of intersection, namely (10, 10) and (6, 6).

<u>Team Round</u>

1. First, since $y = x^2 - 4x - 12 = (x-6)(x+2)$ the x-intercepts of the parabola are $(-2,0)$ and $(6,0)$. Next, complete the square on $x^2 - 4x + y^2 - 8y + 10 = 0$: $x^2 - 4x + 4 + y^2 - 8y + 16 = -10 + 4 + 16$ $\Rightarrow (x-2)^2 + (y-4)^2 = 10 \Rightarrow$ the center of this conic (a circle) is $(2,4)$. So, the unknown parabola passes through the points $(-2,0)$, $(6,0)$, and $(2,4)$. Substituting these points into $y = ax^2 + bx + c$ we have $4a - 2b + c = 0$, $36a + 6b + c = 0$, and $4a + 2b + c = 4$. Subtracting the first equation from the last equation gives $4b = 4 \Rightarrow b = 1$. So, with the first two equations we have $4a + c = 2$ and $36a + c = -6$. Subtracting these equations gives $32a = -8 \Rightarrow a = -\frac{1}{4}$ and therefore $4\left(-\frac{1}{4}\right) + c = 2 \Rightarrow c = 3$. Therefore, the equation of the parabola is $y = -\frac{1}{4}x^2 + x + 3 \Rightarrow a + b + c = -\frac{1}{4} + 3 + 1 = 3\frac{3}{4}$.

2. Since X is the hundreds digit and Y is the tens digit, the numerical value of the number written as XYZ is $100X + 10Y + Z$. The expression to be minimized is therefore $\frac{100X + 10Y + Z}{X + Y + Z}$. This expression can be re-written: $\frac{100X + 10Y + Z}{X + Y + Z} = \frac{10X + 10Y + 10Z}{X + Y + Z} + \frac{90X - 9Z}{X + Y + Z} = 10 + 9 \cdot \frac{10X - Z}{X + Y + Z}$. From this, we must find the smallest value for the numerator and the largest value of the denominator. The smallest value of the numerator occurs when X = 1 and Z = 9. Then, the largest value of the denominator occurs when Y = 8 (since Y cannot also equal 9). Therefore, the smallest value of the expression is $10 + 9 \cdot \frac{1}{18} = 10.5$.

3. First, $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = (6 \cdot 2) \cdot (4 \cdot 3) \cdot 5 = 144 \cdot 5$. Hence, 144 is a factor of 144!. Further, 144 is a factor of any factorial greater than 7! since $7! = 7 \cdot 6!$. Thus, 144 is a factor of every term starting from $6! + 7! + 8! + 9! + 10! + 11!$. There are 95 terms. It remains to see if 144 is a factor of any of the other terms. Clearly the first two terms can be eliminated. Next, $3! + 4! + 5! = 6 + 24 + 120 = 150 \Rightarrow 144$ is not a factor of the third term. Next, $4! + 5! + 6! + 7! = 24 + 120 + 720 + 5040 = 5904 = 144 \cdot 41 \Rightarrow 144$ is a factor of the fourth term. Finally, $5! + 6! + 7! + 8! + 9! = 409080 = 144(2840.8333...) \Rightarrow 144$ is not a factor of the fifth term. Hence there are a total of 96 terms for which 144 is a factor.

4. There are no values greater than 17 because any value greater than 17 would have to be prime and a multiple of 3 which cannot happen. Next, 3 is value less than 17 that is prime, odd, and a multiple of 3. So, 3 works. The other values that work must be in the interval $4 \le n \le 17$. Systematic checking of the numbers in this interval yields 7, 9, 11, 13 and 15. Therefore, the sum is $3 + 7 + 9 + 11 + 13 + 15 = 58$.

5. The sum of the angles of a pentagon is $540°$. If the smallest angle is p and the common difference between the angles is d, then the sum of the angle is also $p + (p+d) + (p+2d) + (p+3d) + (p+4d) = 5p + 10d$. So, $5p + 10d = 540 \Rightarrow p + 2d = 108 \Rightarrow p = 2(54 - d)$. For a convex pentagon we also require that the largest angle be less than $180°$ which means that $p + 4d < 180$. Since $p = 2(54 - d)$ it follows that

$2(54-d)+4d<180 \Rightarrow d<36$. Since all of the angles of the pentagon must be different, d can be any integer from 1 to 35 $\Rightarrow$ there are 35 possibilities for d, each of which gives a unique value for p.

6. Apart from checking a systematic list of Pythagorean triples that contain 20, we can use "Euclid's Formula." Euclid's formula generates Pythagorean triples (a, b, c) given an arbitrary pair of positive integers m and n with $m > n$. The formula states that the integers $a=m^2-n^2$, $b=2mn$, $c=m^2+n^2$ form a Pythagorean triple. If $a=20$, then we have $m^2-n^2=(m-n)(m+n)=20 \Rightarrow m+n=10$ and $m-n=2 \Rightarrow m=6$ and $n=4$. This yields the Pythagorean triple (20, 48, 52). Next, if $b=20$, then $2mn=20 \Rightarrow mn=10 \Rightarrow m=10$ and $n=1$, or $m=5$ and $n=2$. This yields two triples: (20, 99, 101) and (20, 21, 29). Finally, if $c=20$, then $m^2+n^2=20 \Rightarrow m=4$ and $n=2$. This yields the triple (12, 16, 20). Therefore, there are 4 possible triangles satisfying the given conditions.

7. Note that $|2-x|$ is the distance from 2 to x, $|x-y|$ is the distance from x to y and $|y-2012|$ is the distance from y to 2012. Therefore, the sum is the distance traveled from 2 to x to y to 2012. The shortest this distance can be is the distance from 2 to 2012 = 2010.

8. Careful trial-and-error or judicial use of a graphing calculator yields $f^{-1}(2.5)=1.5$ since $f(1.5)=1.5+\lfloor 1.5 \rfloor = 1.5+1=2.5$.

9. There are $5\cdot 5\cdot 5-1=124$ three-digit arrangements that contain all even digits (subtract 1 because "000" would be counted as 0, and something like "024" or "006" are allowed because they count as the numbers 24 and 6, respectively). Now, subtract 124 from 999 and add 1 to include 1000 $\Rightarrow$ there are $999-124+1=876$ such numbers.